

A NOTE ON AMENABILITY OF LOCALLY COMPACT QUANTUM GROUPS

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ABSTRACT. In this short note we introduce a notion called “quantum injectivity” of locally compact quantum groups, and prove that it is equivalent to amenability of the dual. Particularly, this provides a new characterization of amenability of locally compact groups.

INTRODUCTION

In abstract harmonic analysis, the connection between amenability of a locally compact group G and injectivity of its group von Neumann algebra $VN(G)$ is well known: while the former always implies the latter, the converse is generally not true (cf. Connes [3, Corollary 7]); it is, however, true for all discrete groups (and more generally, all inner-amenable groups). See [12, 14] for full details. The notions of amenability and injectivity being of fundamental importance, it is natural to ask whether the relations between them carry to the framework of locally compact *quantum* groups (in the sense of Kustermans and Vaes). Partial answers have been known for some time. Let \mathbb{G} be a locally compact quantum group. If \mathbb{G} is amenable then $L^\infty(\hat{\mathbb{G}})$ is an injective von Neumann algebra (see Enock and Schwartz [8] for Kac algebras, Bédos and Tuset [2] and Doplicher, Longo, Roberts and Zsidó [6] for the general case). Conversely, Ruan [13] proved that if \mathbb{G} is a discrete Kac algebra, then injectivity of $L^\infty(\hat{\mathbb{G}})$ entails amenability of \mathbb{G} . Nevertheless, it is still an open question whether this holds for general discrete quantum groups, not necessarily of Kac type.

While attempting to tackle this problem, we found a quantum analogue of injectivity of von Neumann algebras, which we call “quantum injectivity”. Using the structure theory of completely bounded module maps, we prove that quantum injectivity is equivalent to amenability of the dual in *all* locally compact quantum groups (not necessarily discrete or of Kac type). Particularly, we obtain a new characterization of amenability of locally compact groups. Whether this technique can be used to solve the open question mentioned above is still yet to be seen.

1. PRELIMINARIES

The theory of locally compact quantum groups is by now well established. In this short note we only give the necessary definitions and facts, and refer the reader to Kustermans and Vaes [10, 11] for full details.

Definition 1. A *locally compact quantum group* (LCQG) in the von Neumann algebraic setting is a pair $\mathbb{G} = (M, \Delta)$ such that:

- (1) M is a von Neumann algebra

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- (2) $\Delta : M \rightarrow M \otimes M$ is a co-multiplication, i.e., a unital normal $*$ -homomorphism which satisfies the co-associativity condition:

$$(\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta$$

- (3) There exist normal, semi-finite and faithful weights φ, ψ , called the left and right Haar weights, that are left and right invariant (respectively) in the sense that:

- a. $\varphi((\omega \otimes \text{id})\Delta(x)) = \omega(\mathbf{1})\varphi(x)$ for all $\omega \in M_*^+$ and $x \in M^+$ such that $\varphi(x) < \infty$
- b. $\psi((\text{id} \otimes \omega)\Delta(x)) = \omega(\mathbf{1})\psi(x)$ for all $\omega \in M_*^+$ and $x \in M^+$ such that $\psi(x) < \infty$.

Following the standard convention, we use the notations $L^\infty(\mathbb{G})$, $L^1(\mathbb{G})$ and $L^2(\mathbb{G})$ for M , M_* and the Hilbert space obtained in the GNS construction of (M, φ) , respectively. The canonical injection $\mathcal{N}_\varphi \rightarrow L^2(\mathbb{G})$ is denoted by Λ_φ .

Every locally compact quantum group \mathbb{G} admits a dual LCQG, denoted by $\hat{\mathbb{G}}$. This duality extends the classical one for locally compact abelian groups, and features a Pontryagin-like theorem. Importantly, $L^\infty(\hat{\mathbb{G}})$ can be realized canonically over $L^2(\mathbb{G})$. We say that \mathbb{G} is *compact* if $\varphi(\mathbf{1}) < \infty$ (see Woronowicz [19] for the original definition, independent of the Kustermans–Vaes axioms), and *discrete* if $\hat{\mathbb{G}}$ is compact (cf. [15]). One of the basic objects is the left regular co-representation: it is the multiplicative unitary $W \in L^\infty(\mathbb{G}) \otimes L^\infty(\hat{\mathbb{G}})$ given by

$$W^*(\Lambda_\varphi(a) \otimes \Lambda_\varphi(b)) = \Lambda_{\varphi \otimes \varphi}(\Delta(b)(a \otimes \mathbf{1})) \quad \text{for all } a, b \in \mathcal{N}_\varphi. \quad (1)$$

It implements the co-multiplication as follows:

$$\Delta(x) = W^*(\mathbf{1} \otimes x)W \quad \text{for all } x \in L^\infty(\mathbb{G}). \quad (2)$$

Similarly, there is the right regular co-representation, which is a unitary $V \in L^\infty(\hat{\mathbb{G}})' \otimes L^\infty(\mathbb{G})$ satisfying $\Delta(x) = V(x \otimes \mathbf{1})V^*$ for all $x \in L^\infty(\mathbb{G})$.

Definition 2 ([8, 5, 2]). A LCQG \mathbb{G} is called *amenable* if it admits a left-invariant mean, that is, a state $m \in L^\infty(\mathbb{G})^*$ with $m((\omega \otimes \text{id})\Delta(x)) = \omega(\mathbf{1})m(x)$ for all $x \in L^\infty(\mathbb{G})$, $\omega \in L^1(\mathbb{G})$.

2. QUANTUM INJECTIVITY

Our main result is the following.

Theorem 3. *Let \mathbb{G} be a LCQG. The following conditions are equivalent:*

- (1) \mathbb{G} is amenable
- (2) there is a conditional expectation of $B(L^2(\mathbb{G}))$ onto $L^\infty(\hat{\mathbb{G}})$ that maps $L^\infty(\mathbb{G})$ to $\mathbb{C}\mathbf{1}$
- (3) there is a conditional expectation of $B(L^2(\mathbb{G}))$ onto $L^\infty(\hat{\mathbb{G}})$ that maps $L^\infty(\mathbb{G})$ to $\text{Center}(L^\infty(\hat{\mathbb{G}}))$.

We have recently found out that after we had discovered Theorem 3, Crann and Neufang [4] proved a similar result (from a different perspective), using essentially the same methods.

Definition 4. Let \mathbb{G} be a LCQG. We say that \mathbb{G} is *quantum injective* if there exists a conditional expectation of $B(L^2(\mathbb{G}))$ onto $L^\infty(\mathbb{G})$ that maps $L^\infty(\hat{\mathbb{G}})$ to $\text{Center}(L^\infty(\mathbb{G}))$.

Now Theorem 3 simply says that \mathbb{G} is amenable $\iff \hat{\mathbb{G}}$ is quantum injective.

Remark 5. A LCQG \mathbb{G} is quantum injective if and only if there exists a conditional expectation of $B(L^2(\mathbb{G}))$ onto either $L^\infty(\mathbb{G})$ or $L^\infty(\mathbb{G})'$ that maps either $L^\infty(\hat{\mathbb{G}})$ or $L^\infty(\hat{\mathbb{G}})'$ to $\text{Center}(L^\infty(\mathbb{G}))$ (four equivalent versions). This is a result of the relations $JL^\infty(\mathbb{G})J = L^\infty(\mathbb{G})'$, $JL^\infty(\hat{\mathbb{G}})J = L^\infty(\hat{\mathbb{G}})$, $\hat{J}L^\infty(\hat{\mathbb{G}})\hat{J} = L^\infty(\hat{\mathbb{G}})'$ and $\hat{J}L^\infty(\mathbb{G})\hat{J} = L^\infty(\mathbb{G})$ ([11]).

If R is a von Neumann algebra over a Hilbert space \mathcal{H} , we let $\mathcal{CB}_R(B(\mathcal{H}))$ stand for all completely bounded, R -module maps over $B(\mathcal{H})$. This space always contains the *elementary operators*, namely the ones of the form $x \mapsto \sum_{i=1}^n a'_i x b'_i$, where the a'_i, b'_i belong to R' .

For a LCQG \mathbb{G} , the co-multiplication Δ extends to a normal homomorphism $\bar{\Delta} : B(L^2(\mathbb{G})) \rightarrow L^\infty(\mathbb{G}) \otimes B(L^2(\mathbb{G}))$ given by $\bar{\Delta}(x) := W^*(\mathbb{1} \otimes x)W$ for $x \in B(L^2(\mathbb{G}))$.

Theorem 6. *Let \mathbb{G} be a LCQG. If $E \in \mathcal{CB}_{L^\infty(\hat{\mathbb{G}})}(B(L^2(\mathbb{G})))$, then*

$$(\omega \otimes \text{id})\bar{\Delta}(E(x)) = E((\omega \otimes \text{id})\bar{\Delta}(x)) \quad (\forall x \in B(L^2(\mathbb{G})), \omega \in B(L^2(\mathbb{G}))_*).$$

The theorem would be an easy consequence of (2) if we knew that E was *normal*, but that is rarely true (cf. [17, §V.2, Ex. 8(b)]).

Proof of Theorem 6. On account of the assumption that $E \in \mathcal{CB}_{L^\infty(\hat{\mathbb{G}})}(B(L^2(\mathbb{G})))$ there exists by [7, Theorem 2.5] a net (E_i) of elementary operators with coefficients in $L^\infty(\hat{\mathbb{G}})'$ that converges point-ultraweakly¹ to E . Let $\rho, \omega \in B(L^2(\mathbb{G}))_*$ be given. Fix an index i . Let $\hat{a}'_j, \hat{b}'_j \in L^\infty(\hat{\mathbb{G}})'$, $j = 1, \dots, n$, be such that $E_i x = \sum_{j=1}^n \hat{a}'_j x \hat{b}'_j$ for all $x \in B(L^2(\mathbb{G}))$. Hence, for all $x \in B(L^2(\mathbb{G}))$,

$$(\omega \otimes \rho)\bar{\Delta}(E_i x) = (\omega \otimes \rho)\left(\sum_{j=1}^n W^*(\mathbb{1} \otimes \hat{a}'_j x \hat{b}'_j)W\right).$$

Since $W \in L^\infty(\mathbb{G}) \otimes L^\infty(\hat{\mathbb{G}})$, we obtain

$$\begin{aligned} (\omega \otimes \rho)\bar{\Delta}(E_i x) &= (\omega \otimes \rho)\left(\sum_{j=1}^n (\mathbb{1} \otimes \hat{a}'_j)W^*(\mathbb{1} \otimes x)W(\mathbb{1} \otimes \hat{b}'_j)\right) \\ &= (\omega \otimes \rho)\left(\sum_{j=1}^n (\mathbb{1} \otimes \hat{a}'_j)\bar{\Delta}(x)(\mathbb{1} \otimes \hat{b}'_j)\right) \\ &= (\rho \circ E_i)((\omega \otimes \text{id})\bar{\Delta}(x)). \end{aligned}$$

In conclusion,

$$(\omega \otimes \rho)\bar{\Delta}(E_i x) = (\rho \circ E_i)((\omega \otimes \text{id})\bar{\Delta}(x)) \tag{3}$$

for all i . Since $\bar{\Delta}$ is normal, the limit of the left-hand side of (3) with respect to i is $(\omega \otimes \rho)\bar{\Delta}(Ex) = \rho((\omega \otimes \text{id})\bar{\Delta}(Ex))$, and that of the right-hand side is $(\rho \circ E)((\omega \otimes \text{id})\bar{\Delta}(x))$. The foregoing being true for all $\rho \in B(L^2(\mathbb{G}))_*$, we are done. \square

Proof of Theorem 3. (3) \implies (1). Let E be a conditional expectation of $B(L^2(\mathbb{G}))$ onto $L^\infty(\hat{\mathbb{G}})$ that maps $L^\infty(\mathbb{G})$ to $\text{Center}(L^\infty(\hat{\mathbb{G}}))$. Since E is a completely positive $L^\infty(\hat{\mathbb{G}})$ -bimodule map, it satisfies the conditions of Theorem 6. Fix a state $\rho \in L^\infty(\hat{\mathbb{G}})^*$, and define $m \in L^\infty(\mathbb{G})^*$ by $m := \rho \circ E|_{L^\infty(\mathbb{G})}$. So m is a state since E is a conditional expectation. Moreover, for all

¹Note that in $\mathcal{CB}_{L^\infty(\hat{\mathbb{G}})}(B(L^2(\mathbb{G})))$, w^* -convergence implies point-ultraweak convergence by [7, Lemma 2.4].

$x \in L^\infty(\mathbb{G})$ and $\omega \in L^1(\mathbb{G})$ we have from Theorem 6:

$$\begin{aligned} m((\omega \otimes \text{id})\Delta(x)) &= \rho \circ E((\omega \otimes \text{id})\Delta(x)) = \rho[(\omega \otimes \text{id})(W^*(1 \otimes E(x))W)] \\ &= \rho[(\omega \otimes \text{id})(1 \otimes E(x))] = \omega(1)\rho(E(x)) = \omega(1)m(x) \end{aligned}$$

(because $W \in L^\infty(\mathbb{G}) \otimes L^\infty(\hat{\mathbb{G}})$). Thus m is a left-invariant mean of \mathbb{G} , which is therefore amenable.

Evidently (2) \implies (3). The implication (1) \implies (2) was established long ago in, e.g., [2, Theorem 3.3] (see Remark 5), but without indicating that $E(L^\infty(\mathbb{G})) = \mathbb{C}1$. For completeness, we sketch the argument. Suppose that m is a left invariant mean on $L^\infty(\mathbb{G})$, and denote by V the right regular co-representation of \mathbb{G} . For $x \in B(L^2(\mathbb{G}))$, define $E(x) \in B(L^2(\mathbb{G}))$ by

$$\omega(E(x)) = m((\omega \otimes \text{id})(V(x \otimes 1)V^*)) \quad (\forall \omega \in B(L^2(\mathbb{G}))_*).$$

Then $E : B(L^2(\mathbb{G})) \rightarrow B(L^2(\mathbb{G}))$ is clearly unital and positive. If $x \in L^\infty(\hat{\mathbb{G}})$, then as m is a mean and $V \in L^\infty(\hat{\mathbb{G}})' \otimes L^\infty(\mathbb{G})$, we have $\omega(E(x)) = m((\omega \otimes \text{id})(x \otimes 1)) = \omega(x)$ for all $\omega \in B(L^2(\mathbb{G}))_*$, so that $E(x) = x$. We have to show that the range of E is precisely $L^\infty(\hat{\mathbb{G}}) = L^\infty(\hat{\mathbb{G}})'' = \{y \in B(L^2(\mathbb{G})) : V(y \otimes 1)V^* = y \otimes 1\}$ (cf. [11], proof of Proposition 4.2). To this end, let $\omega, \rho \in B(L^2(\mathbb{G}))_*$ be given, and define $\gamma \in B(L^2(\mathbb{G}))_*$ by $\gamma(x) := (\omega \otimes \rho)[V(x \otimes 1)V^*]$. For $x \in B(L^2(\mathbb{G}))$ we obtain

$$(\omega \otimes \rho)[V(E(x) \otimes 1)V^*] = \gamma(E(x)) = m((\gamma \otimes \text{id})(V(x \otimes 1)V^*)). \quad (4)$$

The identity $V_{12}V_{13} = (\text{id} \otimes \Delta)(V)$ shows that

$$(\gamma \otimes \text{id})(V(x \otimes 1)V^*) = (\rho \otimes \text{id})\Delta[(\omega \otimes \text{id})(V(x \otimes 1)V^*)]. \quad (5)$$

By (4), (5) and left invariance of m we have $(\omega \otimes \rho)[V(E(x) \otimes 1)V^*] = \rho(1)\omega(E(x))$. Since ω and ρ were arbitrary, we conclude that $V(E(x) \otimes 1)V^* = E(x) \otimes 1$ and hence $E(x) \in L^\infty(\hat{\mathbb{G}})$ indeed. Finally, if $x \in L^\infty(\mathbb{G})$, then $V(x \otimes 1)V^* = \Delta(x)$, and the left invariance of m implies that $\omega(E(x)) = m((\omega|_{L^\infty(\mathbb{G})} \otimes \text{id})\Delta(x)) = \omega|_{L^\infty(\mathbb{G})}(1)m(x) = \omega(m(x)1)$ for all $\omega \in B(L^2(\mathbb{G}))_*$, so that $E(x) = m(x)1$. \square

As a result, we have the following new characterization of amenability of groups.

Corollary 7. *Let G be a locally compact group. Then G is amenable \iff there is a conditional expectation of $B(L^2(G))$ onto $\text{VN}(G)$ mapping $L^\infty(G)$ to $\text{Center}(\text{VN}(G))$ (or to the scalars).*

When \mathbb{G} is a discrete quantum group of Kac type, Ruan [13] proved that injectivity of $L^\infty(\hat{\mathbb{G}})$ implies amenability of \mathbb{G} by proving directly that $\hat{\mathbb{G}}$ is co-amenable (in Ruan's nomenclature: \mathbb{G} is strongly [Voiculescu] amenable). Using Theorem 6, we give a short, direct proof of this fact (compare [1, Theorem 4.9]), in the same spirit as the proof of the group case. Recall that the left regular co-representation \hat{W} of $\hat{\mathbb{G}}$ is equal to $\sigma(W)^*$, where σ is the flip map $x \otimes y \mapsto y \otimes x$ on $B(L^2(\mathbb{G})) \otimes B(L^2(\mathbb{G}))$.

Corollary 8. *If \mathbb{G} is a discrete quantum group of Kac type and $L^\infty(\hat{\mathbb{G}})$ is injective, then \mathbb{G} is amenable.*

Proof. Since $\hat{\mathbb{G}}$ is compact and of Kac type, its Haar state $\hat{\varphi}$ is a trace. Let E be a conditional expectation from $B(L^2(\mathbb{G}))$ onto $L^\infty(\hat{\mathbb{G}})$. Define $m := \hat{\varphi} \circ E|_{L^\infty(\mathbb{G})}$. If $x \in L^\infty(\mathbb{G})$ and $\hat{a}, \hat{b} \in L^\infty(\hat{\mathbb{G}})$, then for $\omega := \omega_{\Lambda_{\hat{\varphi}}(\hat{a}), \Lambda_{\hat{\varphi}}(\hat{b})}$ we have from Theorem 6 and the dual version of (1):

$$\begin{aligned} m((\omega \otimes \text{id})\Delta(x)) &= (\omega \otimes \hat{\varphi})\overline{\Delta}(E(x)) = (\hat{\varphi} \otimes \omega)(\hat{W}(E(x) \otimes \mathbf{1})\hat{W}^*) \\ &= \left\langle \hat{W}(E(x) \otimes \mathbf{1})\hat{W}^*(\Lambda_{\hat{\varphi}}(\mathbf{1}) \otimes \Lambda_{\hat{\varphi}}(\hat{a})), \Lambda_{\hat{\varphi}}(\mathbf{1}) \otimes \Lambda_{\hat{\varphi}}(\hat{b}) \right\rangle \\ &= \left\langle (E(x) \otimes \mathbf{1})\hat{W}^*(\Lambda_{\hat{\varphi}}(\mathbf{1}) \otimes \Lambda_{\hat{\varphi}}(\hat{a})), \hat{W}^*(\Lambda_{\hat{\varphi}}(\mathbf{1}) \otimes \Lambda_{\hat{\varphi}}(\hat{b})) \right\rangle \\ &= (\hat{\varphi} \otimes \hat{\varphi})(\hat{\Delta}(\hat{b}^*)(E(x) \otimes \mathbf{1})\hat{\Delta}(\hat{a})). \end{aligned}$$

By the traciality and invariance of $\hat{\varphi}$ we deduce that

$$m((\omega \otimes \text{id})\Delta(x)) = (\hat{\varphi} \otimes \hat{\varphi})((E(x) \otimes \mathbf{1})\hat{\Delta}(\hat{a}\hat{b}^*)) = \hat{\varphi}(E(x))\hat{\varphi}(\hat{a}\hat{b}^*) = \omega(\mathbf{1})m(x).$$

Therefore m is a left invariant mean, and \mathbb{G} is amenable. \square

We conclude with the original open question that has been the motivation for this note. Since a compact quantum group is of Kac type if and only if its underlying von Neumann algebra is finite (see [16, Remark A.2]; also compare Fima [9, Theorem 8]), this question is of interest only when $L^\infty(\hat{\mathbb{G}})$ is *infinite* by the last corollary.

Conjecture. *If \mathbb{G} is a discrete quantum group such that $L^\infty(\hat{\mathbb{G}})$ is injective, then \mathbb{G} is amenable (and hence $\hat{\mathbb{G}}$ is co-amenable by Tomatsu [18]).*

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